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On the equivalence of the Flügge-Koiter buckling equations to those recently derived in current literature*

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Abstract. The main goal of this work is to prove the equivalence of various formulations existing in the current literature which give the governing equations of small displacements superimposed over initial static deformations of thin cylindrical shells. It is shown that the main difference comes from the definition of incremental stress resultants and the dependence of elastic coefficients on initial deformation. When the functional dependence of elastic coefficients on initial deformation is taken into consideration, it is shown that all the derivations are equivalent to each other.

Key words: Flügge-Koiter, buckling, cylindrical shells, elasticity

1. Introduction

Researchers working in the area of wave propagation in prestressed circular cylindrical thin tubes filled with a viscous or inviscid fluid use the field equations of the so-called 'theory of small displacements superimposed on large initial static deformations'. Worthy of mentioning among such theories are the buckling equations of thin cylindrical shells by Flügge-Koiter (Flügge [1]), the mechanics of incremental deformations by Biot [2] and the equations of motion of fluid-filled prestressed elastic tubes by Atabek and Lew [3] and ones more recently derived by various researchers employing the tools of modern continuum mechanics (see, for instance, Rachev [4] and Demiray [5]). The derivation of the buckling equations of Flügge and Koiter is essentially based on equilibrium equations referred to the intermediate configuration and the use of the second type of the Piola-Kirchoff stress tensor to define the stress resultants. On the other hand, Atabek and Lew [3] derived their equations referring to the final configuration and used the Cauchy stress tensor in their definition of stress resultants. Of course, the incremental stress resultants and the equations of motion presented by these researchers appear to be different from each other. In essence, they should not be different when they are expressed in terms of displacement components. As I see it, the fundamental mistake made by these researchers is that, in writing the linear constitutive relations for incremental stress resultants in terms of small strains, both groups assumed the coefficients of their linear expressions to be material constants. In reality, these coefficients are not simple material constants, but rather are coefficients which depend both on the material properties and the initial deformation. If the functional dependence of these coefficients on initial deformation is known, (see, Rachev [4] and Demiray [5]) then it will be an easy exercise to convert one set into the other and see that these two sets of equations are equivalent. Because of these conceptual mistakes, researchers in the field are often accused of using the wrong set of

* This paper is dedicated to Professor F. Ziegler for his 60th birthday.

equations by proponents of the other set (see, for instance, Kuiken [6] and Hart and Shi [7]). We were not aware of such a formal difference between these two derivations until we were faced with criticism for the equations we used in our research.

In the present work, we shall prove that these two formulations are equivalent by using the tools of modern continuum mechanics. For this purpose, by employing the theory of so-called 'small deformations superimposed on large initial static deformations' we re-derive these two sets of equations for prestressed cylindrical shells subjected to axially symmetric time-dependent incremental displacements. In the Flügge-Koiter formulation we utilize the second type of Piola-Kirchhoff incremental stress tensor, whereas in the other derivations, the incremental Cauchy tensor is employed. Using the transformation rule between these two types of incremental stress tensors, we then show that these two formulations are equivalent. It is observed that the main difference between these derivations comes from the definition of incremental stress resultants and the dependence of the coefficients of these incremental stresses on the initial deformations. By utilizing a strain-energy density function proposed by the author (Demiray [8]) for soft biological tissues, we show that for small initial deformations these incremental coefficients may be treated as constants, whereas for large initial deformations they are variable. This property of the coefficients should be taken into account when studying the propagation of waves in large blood vessels.

2. Derivation of basic equations

In this section we shall derive the governing equations of an initially stressed cylindrical thin shell when small dynamical displacements are superimposed on the initial deformations. For this purpose, we shall utilize the theory of so-called 'small deformations superimposed on large initial static deformations' and derive the governing equations of these two seemingly different formulations and then prove that these two approaches are equivalent.

Let us consider a circular cylindrical membrane of radius R_0, subjected to an inner pressure P_i, and a uniformly distributed axial force N. Upon application of these symmetrical forces, the membrane will deform into another cylindrical shell with radius r_0 and with membrane forces N_z^0 and N_θ^0 in the axial and circumferential directions, respectively. Onto this initial static deformation, we shall superimpose a symmetrical and time-dependent displacement with components $u(z,t)$ and $w(z,t)$ in the radial and axial directions, respectively. Then the position vector of a representative point (r_0, θ, z), will be as follows

$$\mathbf{r} = (r_0 + u)\mathbf{e}_r + (z + w)\mathbf{e}_z, \tag{1}$$

where $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$ are the unit base vectors in cylindrical polar coordinates. In the course of such a deformation, a unit line element along the generator (or unit vector $\mathbf{e}_z$) will deform into the vector $\mathbf{t}_z$ defined by

$$\mathbf{t}_z = \frac{\partial \mathbf{r}}{\partial z} = u_{,z}\,\mathbf{e}_r + (1 + w_{,z})\mathbf{e}_z, \tag{2}$$

where for brevity we defined $(\ )_{,z} = \frac{\partial(\ )}{\partial z}$. Similarly the tangent vector $\mathbf{e}_\theta$ will deform into

$$\mathbf{t}_\theta = \frac{1}{r_0}\frac{\partial \mathbf{r}}{\partial \theta} = (1 + (u/r_0))\mathbf{e}_\theta. \tag{3}$$

The unit vectors along these vectors shall be denoted by $\mathbf{t}$ and $\mathbf{e}_\theta$, respectively.

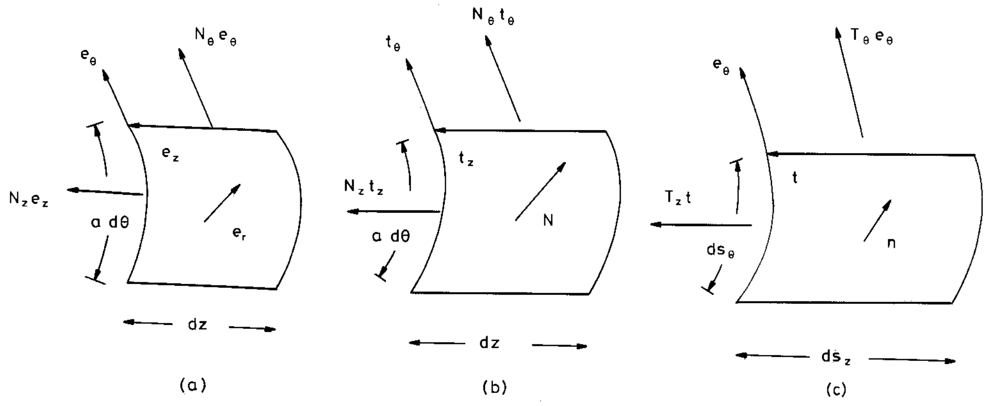

Figure 1. Small shell elements with stress resultants

Now, let us consider a small cylindrical shell element located between the planes $z =$ const, $z + \mathrm{d}z =$ const, $\theta =$ const and $\theta + \mathrm{d}\theta =$ const (Figure 1(a)). After superimposing the displacement-field components u and w on the given initial static deformation, we observe that this small element deforms into the configuration given in Figure 1(c), with side lengths $\mathrm{d}s_\theta$ and $\mathrm{d}s_z$, where

$$\mathrm{d}s_\theta = (1 + (u/r_0))r_0\,\mathrm{d}\theta, \qquad \mathrm{d}s_z = \wedge_z\,\mathrm{d}z, \tag{4}$$

with

$$\wedge_z = [u_{,z}^2 + (1 + w_{,z})^2]^{1/2}. \tag{5}$$

In what follows we shall present two different formulations for the equations of motion governing the tube element.

2.1. Flügge-Koiter Formulation

Let us denote the total membrane force resultants acting on the unit length of the intermediately deformed configuration by N_θ and N_z, or, in vector form by $N_\theta\mathbf{e}_\theta$ and $N_z\mathbf{e}_z$. When we let the body undergo deformation, these force vectors will deform and take one of the configurations described in Figures 1(b) and 1(c). In configuration 1(b), we shrink the deformed geometry into the original dimension with a similar geometry, but rotate and stretch the base vectors $\mathbf{e}_\theta$ and $\mathbf{e}_z$ into $\mathbf{t}_\theta$ and $\mathbf{t}_z$, respectively. As will be shown in the subsequent sections, the stress resultants used by Flügge and Koiter may be expressed in terms of the total second type of the Piola-Kirchhoff stress tensor S_{kl} referred to the intermediately deformed configuration. For instance, the stress resultant N_θ may be expressed as $N_\theta = hS_{\theta\theta}$, where h is the thickness in the intermediate configuration and $S_{\theta\theta}$ is the normal component of S_{kl} in the $\mathbf{e}_\theta$ direction. However, the stress resultants for other formulations are expressed in terms of the Cauchy stress tensor t'_{kl}, e.g., $T_\theta = h't'_{\theta\theta}$, where h' is the thickness in the final configuration and $t'_{\theta\theta}$ is the circumferential component of the Cauchy stress tensor referred to the final configuration.

From the inspection of Figures 1(b) and 1(c), it can be shown that these stress resultants are related to each other by

$$T_\theta = N_\theta(1 + (u/r_0))/\wedge_z \quad \text{and} \quad T_z = N_z \wedge_z /(1 + (u/r_0)). \tag{6}$$

Let the total external force acting on the unit area in the intermediate configuration be expressed as

$$\mathbf{F} = F_N \mathbf{N} + F_t \mathbf{t}_z. \tag{7}$$

Similarly, we denote the force vector acting per unit deformed area by

$$\mathbf{P} = P_n \mathbf{n} + P_t \mathbf{t}, \tag{8}$$

where $\mathbf{N} = \mathbf{t}_\theta \times \mathbf{t}_z$ and $\mathbf{n}$ is the unit vector along $\mathbf{N}$. These two vector fields have the following relation

$$F_N = P_n, \qquad F_t = (1 + (u/r_0))P_t. \tag{9}$$

Referring to Figure 1(b), we may give the total force acting per unit area as follows:

$$\frac{\partial}{\partial z}[N_z \mathbf{t}_z] + \frac{1}{r_0}\frac{\partial}{\partial \theta}[N_\theta \mathbf{t}_\theta] + F_N \mathbf{N} + F_t \mathbf{t}_z. \tag{10}$$

This force should be equal to the mass times the acceleration of the element with unit area given by

$$\rho h \left(\frac{\partial^2 u}{\partial t^2}\mathbf{e}_r + \frac{\partial^2 w}{\partial t^2}\mathbf{e}_z \right). \tag{11}$$

Equating the expressions (10) and (11), we have

$$\frac{\partial}{\partial z}[N_z \mathbf{t}_z] + \frac{1}{r_0}\frac{\partial}{\partial \theta}[N_\theta \mathbf{t}_\theta] + F_N \mathbf{N} + F_t \mathbf{t}_z = \rho h \left(\frac{\partial^2 u}{\partial t^2}\mathbf{e}_r + \frac{\partial^2 w}{\partial t^2}\mathbf{e}_z \right). \tag{12}$$

Introducing the explicit expressions for $\mathbf{t}_\theta$ and $\mathbf{t}_z$ into (12), we may obtain the equations of motion in component form. However, in what follows we shall give the linearized governing equations.

2.1.1. *Linearization around the pre-deformed state*

For future purposes, in this subsection we shall give the linearized equations of motion. Therefore, we set

$$N_z = N_z^0 + \bar{N}_z, \qquad N_\theta = N_\theta^0 + \bar{N}_\theta, \qquad F_N = P_i + \bar{F}_N, \qquad F_t = 0 + \bar{F}_t, \tag{13}$$

where the over-barred quantities stand for increments in the corresponding fields, with similar orders of u and w which are assumed to be small. Introducing (13) into (12) and recalling the expressions for $\mathbf{t}_\theta$ and $\mathbf{t}_z$, we obtain the following linearized equations of motion.

$$\begin{aligned} &N_z^0 \frac{\partial^2 u}{\partial z^2} + \frac{N_\theta^0}{r_0}\frac{\partial w}{\partial z} - \frac{\bar{N}_\theta}{r_0} + \bar{F}_N = \rho h \frac{\partial^2 u}{\partial t^2}, \\ &N_z^0 \frac{\partial^2 w}{\partial z^2} - \frac{N_\theta^0}{r_0}\frac{\partial u}{\partial z} + \frac{\partial \bar{N}_z}{\partial z} + \bar{F}_t = \rho h \frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{14}$$

These equations are exactly the same as those of Kuiken [6] who employed Flügge's [1] buckling equations derived for thin cylindrical shells in studying the problem of wave propagation in prestressed elastic tubes.

2.2. ATABEK–DEMIRAY FORMULATION

This formulation is completely based on the interpretation given in Figure 1(c). In order to write the equations of motions in this configuration, we should first write down the expression of the total force acting on the deformed element. This force may be expressed as

$$\frac{\partial}{\partial z}[T_z \mathbf{t}(r_0+u)]\,\mathrm{d}\theta\,\mathrm{d}z + \frac{\partial}{\partial \theta}[T_\theta \mathbf{e}_\theta \Lambda_z]\,\mathrm{d}\theta\,\mathrm{d}z + (P_n\mathbf{n} + P_t\mathbf{t})(r_0+u)\Lambda_z\,\mathrm{d}\theta\,\mathrm{d}z. \tag{15}$$

This force should be equal to the total mass of the element times the acceleration vector, which is given by

$$\rho' h'(r_0+u)\Lambda_z\,\mathrm{d}\theta\,\mathrm{d}z\left(\frac{\partial^2 u}{\partial t^2}\mathbf{e}_r + \frac{\partial^2 w}{\partial t^2}\mathbf{e}_z\right), \tag{16}$$

where ρ' is the final mass density and h' is the deformed thickness of the membrane. Equating equations (15) and (16), we obtain

$$\left\{\frac{\partial}{\partial z}[T_z(r_0+u)\mathbf{t}] + \frac{\partial}{\partial \theta}[T_\theta \Lambda_z \mathbf{e}_\theta]\right\}[(r_0+u)\Lambda_z]^{-1}$$
$$+(P_n\mathbf{n} + P_t\mathbf{t}) = \rho' h'\left(\frac{\partial^2 u}{\partial t^2}\mathbf{e}_r + \frac{\partial^2 w}{\partial t^2}\mathbf{e}_z\right). \tag{17}$$

Using the explicit expressions for **t** and **n**, we may write these equations in component form. However, for future purposes we need only the linearized equations, so that we shall not list the general equations here.

2.2.1. *Linearization around the pre-deformed state*

In order to be able to compare these two formulations, in this sub-section we shall give the linearized equations of motion, and for this purpose we set

$$\begin{aligned} &T_z = N_z^0 + \bar{\Sigma}_z, \qquad T_\theta = N_\theta^0 + \bar{\Sigma}_\theta, \qquad P_n = P_i + \bar{P}_n, \\ &P_t = 0 + \bar{P}_t, \qquad h' = h + \bar{h}, \qquad \rho' = \rho + \bar{\rho}, \end{aligned} \tag{18}$$

where the over-barred quantities represent increments for this configuration and the order of them will be assumed to be same as the incremental displacement. Introducing (18) into (17) and recalling the expressions of **t** and **n**, we obtain the following linearized equations of motion for this configuration.

$$\begin{aligned} &N_z^0\frac{\partial^2 u}{\partial z^2} + \frac{N_\theta^0 u}{r_0^2} - \frac{\bar{\Sigma}_\theta}{r_0} + \bar{P}_n = \rho h\frac{\partial^2 u}{\partial t^2}, \\ &\left(\frac{N_z^0 - N_\theta^0}{r_0}\right)\frac{\partial u}{\partial z} + \frac{\partial \bar{\Sigma}_z}{\partial z} + \bar{P}_t = \rho h\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{19}$$

Furthermore, by using the relations given in (9), we can show that in the linear case the following equations hold: $\bar{F}_N = \bar{P}_n$ and $\bar{F}_t = \bar{P}_t$. The equations (19) are exactly the same as those obtained by Demiray [5].

2.3. Equivalence of these derivations

Although (14) and (19) seem to differ from each other, in essence, they are equivalent. In order to show the equivalence of these two derivations we refer to the relations given in (6). If we use definitions (13) and (18) and apply the linearization procedure, we obtain.

$$\bar{\Sigma}_\theta = \bar{N}_\theta + N_\theta^0\left(\frac{u}{r_0} - \frac{\partial w}{\partial_z}\right), \qquad \bar{\Sigma}_z = \bar{N}_z + N_z^0\left(\frac{\partial w}{\partial z} - \frac{u}{r_0}\right). \tag{20}$$

Introducing (20) into (19) we have

$$\begin{aligned} &N_z^0\frac{\partial^2 u}{\partial z^2} + \frac{N_\theta^0}{r_0}\frac{\partial w}{\partial z} - \frac{\bar{N}_\theta}{r_0} + \bar{P}_n = \rho h\frac{\partial^2 u}{\partial t^2},\\ &N_z^0\frac{\partial^2 w}{\partial z^2} - \frac{N_\theta^0}{r_0}\frac{\partial u}{\partial z} + \frac{\partial \bar{N}_z}{\partial z} + \bar{P}_t = \rho h\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{21}$$

These equations are exactly the same as those of (14), which are due to Flügge [1].

3. Incremental constitutive relations

One of the main differences between the Flügge-Koiter derivation and the recent ones is in the form of the incremental constitutive relations. In the Flügge-Koiter formulation the incremental stress resultants are expressed in terms of the incremental deformation as

$$\bar{N}_\theta = B_{11}\frac{u}{a} + B_{12}\frac{\partial w}{\partial z}, \qquad \bar{N}_z = B_{21}\frac{u}{a} + B_{22}\frac{\partial w}{\partial z}. \tag{22}$$

Here the coefficients $B_{ij}(i = 1, 2)$ are treated as mere material constants. In reality, these are not material constants, buth rather coefficients that depend on both the material constants and the initial stresses. For engineering materials the initial deformation is not too large; therefore, for such materials the variation of these coefficients with initial deformation may not be important. However, for soft biological materials the initial deformation is very large and the variation of these coefficients with initial deformation may become quite important. Therefore, in studying the problems of wave propagation in initially stressed fluid-filled tubes, as applied to blood-flow problems, the variations of these coefficients should be taken into account. Of course, in order to find the dependence of these coefficients on the initial deformation, one must know the nonlinear stress-strain relations of the material under consideration and should use the theory of so-called ‘small displacements superimposed on large initial static deformations’. In what follows we shall utilize this theory and obtain explicit expressions for these coefficients as functions of the initial deformation.

Let t_{kl}^0 be the Cauchy stress tensor in the statically deformed intermediate configuration and t'_{kl} be the Cauchy stress tensor in the final configuration. The deformation relating the intermediate configuration to the final configuration may be described by

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}(\mathbf{x}, t), \tag{23}$$

where **x** is the position vector of a material point in the intermediate configuration and $\mathbf{u}(\mathbf{x}, t)$ is the superimposed displacement field. The Cauchy stress t'_{kl} is related to the second Piola-Kirchhoff stress tensor S_{kl}, referred to the intermediate configuration, by

$$S_{kl} = j\frac{\partial x_k}{\partial x'_m}\frac{\partial x_l}{\partial x'_n}t'_{mn}, \tag{24}$$

where j is the Jacobian of the motion defined by $j = \det(\frac{\partial x'_k}{\partial x_m})$. If the superposed displacement field is small, we can write

$$\frac{\partial x'_k}{\partial x_m} = \delta_{km} + u_{k;m} \quad \text{and} \quad j = 1 + u_{r;r}. \tag{25}$$

Here the indices following a semi colon denote the covariant (contravariant) differentiation with respect to that indexed coordinate variable and the summation convention applies to repeated indices. Introducing the incremental second Piola-Kirchhoff stress tensor $\bar{S}_{kl}$ and incremental Cauchy stress tensor $\bar{t}_{kl}$ as follows

$$S_{kl} = t^0_{kl} + \bar{S}_{kl}, \qquad t'_{kl} = t^0_{kl} + \bar{t}_{kl} \tag{26}$$

and utilizing the relations (24) and (25), we obtain

$$\bar{S}_{kl} = \bar{t}_{kl} + u_{r;r}t^0_{kl} - u_{k;m}t^0_{ml} - u_{l;m}t^0_{mk}. \tag{27}$$

Soft biological tissue is assumed to be incompressible, which we shall also adopt throughout this study. In this case (27) reduces to

$$\bar{S}_{kl} = \bar{t}_{kl} - u_{k;m}t^0_{ml} - u_{l;m}t^0_{mk}. \tag{28}$$

The total stress resultants for the shell under investigation, referred to the final configuration, is defined by

$$T_\theta = h't'_{\theta\theta}, \qquad T_z = h't'_{zz}. \tag{29}$$

Employing the incompressibility condition, we may express the deformed thickness h' in terms of the thickness h as follows:

$$h' = \frac{h}{(1 + u/r_0)\Lambda_z} \cong h\left(1 - \frac{u}{r_0} - \frac{\partial w}{\partial z}\right). \tag{30}$$

Setting $t'_{\theta\theta} = t^0_{\theta\theta} + \bar{t}_{\theta\theta}, t'_{zz} = t^0_{zz} + \bar{t}_{zz}$ and utilizing (30) in (29), we have

$$\bar{\Sigma}_\theta = h\bar{t}_{\theta\theta} - t^0_{\theta\theta}h\left(\frac{u}{r_0} + \frac{\partial w}{\partial z}\right), \qquad \bar{\Sigma}_z = h\bar{t}_{zz} - t^0_{zz}h\left(\frac{u}{r_0} + \frac{\partial w}{\partial z}\right). \tag{31}$$

Furthermore, from relation (28) we can write

$$\bar{t}_{\theta\theta} = \bar{S}_{\theta\theta} + 2t^0_{\theta\theta}\frac{u}{r_0}, \qquad \bar{t}_{zz} = \bar{S}_{zz} + 2t^0_{zz}\frac{\partial w}{\partial z}. \tag{32}$$

Introducing (32) into (31) and defining the incremental stress resultants $\bar{N}_\theta$ and $\bar{N}_z$ by $\bar{N}_\theta = h\bar{S}_{\theta\theta}, \bar{N}_z = h\bar{S}_{zz}$, we obtain

$$\bar{\Sigma}_\theta = \bar{N}_\theta + N_\theta^0 \left(\frac{u}{r_0} - \frac{\partial w}{\partial z} \right), \qquad \bar{\Sigma}_z = \bar{N}_z + N_z^0 \left(\frac{\partial w}{\partial z} - \frac{u}{r_0} \right). \tag{33}$$

These expressions are exactly the same as those of (20). Therefore, we can state that in the Flügge-Koiter formulation, the total stress resultants are defined in terms of the total Piola-Kirchhoff stress tensor of the second type.

Having arrived at this important conclusion and having obtained relation (28), we may express the coefficients B_{ij} appearing in (22) in terms of the initial deformations. In order to accomplish this, we need the explicit expression of the strain-energy density function. For soft biological tissue the author (Demiray [8]) proposed a strain-energy density function of the form

$$\Sigma = (\beta/2\alpha)\exp[\alpha(I_2 - 3)], \tag{34}$$

where α, β are two material constants and I_2 is the second principal invariant of the Finger deformation tensor.

Employing this type of strain-energy density function, we have found the explicit expressions of $\bar{\Sigma}_\theta$ and $\bar{\Sigma}_z$ as (see Demiray and Antar [9] Equations (2.34) and (2.35))

$$\bar{\Sigma}_\theta = h \left(\alpha_{11}^0 \frac{u}{r_0} + \alpha_{12}^0 \frac{\partial w}{\partial z} \right), \qquad \bar{\Sigma}_z = h \left(\alpha_{21}^0 \frac{u}{r_0} + \alpha_{22}^0 \frac{\partial w}{\partial z} \right), \tag{35}$$

where the coefficients $\alpha_{ij}^0 (i, j = 1, 2)$ are defined by

$$\begin{aligned}
\alpha_{11}^0 &= \beta F(\lambda_\theta)[(\lambda_\theta^2\lambda_z^2 + 3\lambda_\theta^{-2}) + 2\alpha(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2})^2], \\
\alpha_{12}^0 &= \beta F(\lambda_\theta)[(\lambda_\theta^2\lambda_z^2 + \lambda_\theta^{-2}) + 2\alpha(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2})(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2})], \\
\alpha_{21}^0 &= \beta F(\lambda_\theta)[(\lambda_\theta^2\lambda_z^2 + \lambda_z^{-2}) + 2\alpha(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2})(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2})], \\
\alpha_{22}^0 &= \beta F(\lambda_\theta)[(\lambda_\theta^2\lambda_z^2 + 3\lambda_z^{-2}) + 2\alpha(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2})^2],
\end{aligned} \tag{36}$$

where λ_θ and λ_z are the stretch ratios in the circumferential and axial directions, respectively, and the function $F(\lambda_\theta)$ is defined by

$$F(\lambda_\theta) \equiv \exp[\alpha(\lambda_\theta^{-2} + \lambda_z^{-2} + \lambda_\theta^2\lambda_z^2 - 3)]. \tag{37}$$

Introducing (35) into (33) and considering the expressions for N_θ^0 and N_z^0 in terms of the stretch ratios as follows (see, Demiray and Antar [9])

$$N_\theta^0 = h\beta F(\lambda_\theta)(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2}), \qquad N_z^0 = h\beta F(\lambda_\theta)(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2}), \tag{38}$$

we may express the coefficients B_{ij} in the following form:

$$\begin{aligned}
B_{11} &= 2\beta h F(\lambda_\theta)[2\lambda_\theta^{-2} + \alpha(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2})^2], \\
B_{12} = B_{21} &= 2\beta h F(\lambda_\theta)[\lambda_\theta^2\lambda_z^2 + \alpha(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2})(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2})], \\
B_{22} &= 2\beta h F(\lambda_\theta)[2\lambda_z^{-2} + \alpha(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2})^2].
\end{aligned} \tag{39}$$

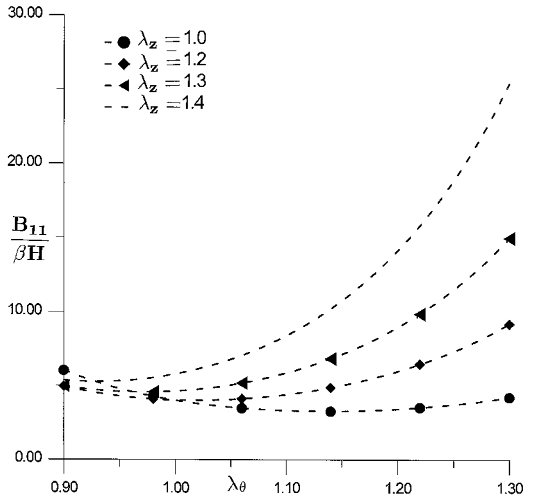

Figure 2. The variation of B_{11} with initial deformation.

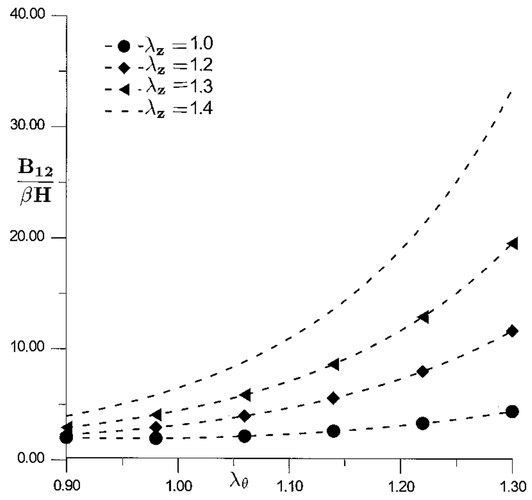

Figure 3. The variation of B_{12} with initial deformation.

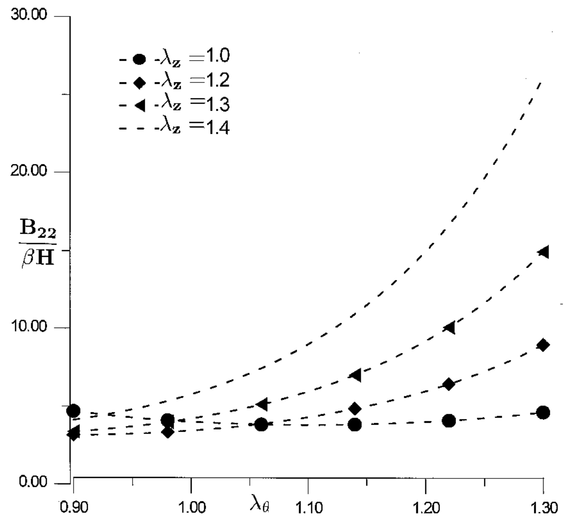

Figure 4. The variation of B_{22} with initial deformation.

These expressions show that the coefficients B_{ij} change with initial deformation. For the details of this derivation the reader is referred to the work of Demiray and Antar [9]. The variations of these coefficient with stretch ratio are evaluated numerically, and the results are depicted in Figures 2–4. These figures indicate that the coefficients $B_{ij}(i, j = 1, 2)$ change considerably with stretch ratio. Therefore, the assumption of constant coefficients is valid for engineering materials and classical buckling problems. However, in applying these incremental laws to the study of wave propagation in arteries, which are initially stressed by a considerable amount, the same assumption does not apply. Hence, in studying such problems, we must take into account the explicit dependence of these coefficients on initial deformation.

4. Concluding remarks

As stated in the Introduction, the objective of this work is to prove the equivalence of two seemingly different formulations existing in the current literature, and used by researchers for the study of wave propagation in prestressed cylindrical thin shells. To this end, by utilizing the theory of so-called 'small deformations superimposed on large initial static deformations', we have re-derived the governing equations of the two different formulations presented by Flügge-Koiter and Atabek-Demiray. In obtaining the equations derived by Flügge and Koiter, we employed the incremental second type of the Piola-Kirchhoff stress tensor referred to the intermediate configuration for defining the incremental stress resultants, whereas for those derived by Atabek and Demiray we utilized the incremental Cauchy stress tensor referred to the final configuration for the definition of incremental stress resultants. Due to these fundamental differences in the definition of stress resultants in these two formulations, the governing equations will look different from each other. Nevertheless, by utilizing the transformation rules between the two incremental stresses and considering the definitions of incremental stress resultants, we proved that these two formulations are equivalent. The most misleading point in these derivations is the assumption of constancy of the incremental elastic coefficients. In reality, they are not constants, but rather are functions of the initial deformation and can be converted from one to the other through the use of a proper transformation rule.

To be more specific, we obtained the explicit expressions for the incremental coefficients of both formulations as functions of the initial deformation, by using a strain-energy density function previously proposed by the author (Demiray [8]) for soft biological tissues. Numerical studies of these coefficients as functions of the initial deformation reveal that, for small initial deformations, these coefficients remain almost constant. Therefore, the assumption of constant elastic coefficients is valid for engineering materials and classical buckling problems. However, for large values of the initial deformation, the variations of these coefficients are considerably larger. Hence, in studying the propagation of harmonic waves in large blood vessels which are subjected to large initial static deformations, the assumption of constant elastic coefficients may lead to incorrect results. Therefore, in studying such problems through the use of these two formulations the explicit dependence of these coefficients on the initial deformation must be taken into account.

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References

1. W. Flügge, *Stress in Shells*. Berlin: Springer Verlag (1973) 525 pp.
2. M. A. Biot, *Mechanics of Incremental Deformations.* New York: John Wiley & Sons, Inc. (1964) 504 pp.
3. H. B. Atabek and H. S. Lew, Wave propagation through a viscous incompressible fluid contained in an initially stressed elastic tube. *Biophys. J.* 6 (1966) 481–503.
4. A. I. Rachev, Effect of transmural pressure and muscular activity on pulse waves in arteries. *J. Biomech. Engng, ASME* 102 (1980) 119–123.
5. H. Demiray, Propagation of harmonic waves in a prestressed elastic tube filled with a viscous fluid. *Bull. Tech. Univ. Istanbul*, Special Issue: Şuhubi and Continuum Mechanics 47 (1994) 151–166.
6. G. D. C. Kuiken, Wave propagation in a thin walled liquid-filled initially stressed tube. *J. Fluid Mech.* 141 (1984) 289–308.
7. V. G. Hart and J. Shi, Effects of initial stretches on waves speeds in thin orthotropic hyperelastic tubes containing fluid flow. *Appl. Math. Modelling* 18 (1994) 198–206.
8. H. Demiray, Stress in ventricular wall. *J. Appl. Mech., ASME* 98 (1976) 194–197.
9. H. Demiray and N. Antar, Harmonic waves in an elastic thin tube filled with a viscous fluid. *J. Engng Math.* (in press).